

# A Poncelet Criterion for special pairs of conics in $PG(2, p)$

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## Abstract

We study Poncelet's Theorem in finite projective coordinate planes over the field  $GF(p)$  and concentrate on a particular pencil of conics. For pairs of such conics we investigate whether we can find polygons with  $n$  sides, which are inscribed in one conic and circumscribed about the other, so-called Poncelet Polygons. By using suitable elements of the dihedral group for these pairs, we prove that the length  $n$  of such Poncelet Polygons is independent of the starting point. In this sense Poncelet's Porism is valid. By using Euler's divisor sum formula for the totient function, we can make a statement about the number of different conic pairs, which carry Poncelet Polygons of length  $n$ . Moreover, we will introduce polynomials whose zeros in  $GF(p)$  yield information about the relation of a given pair of conics. In particular, we can decide for a given integer  $n$ , whether and how we can find Poncelet Polygons for pairs of conics in the given coordinate plane. We will see that this condition is closely connected with the theory of quadratic residues.

## Introduction

In 1813 Jean-Victor Poncelet stated one of the most beautiful results in projective geometry, known as Poncelet's Porism. He proved that for two conics  $C$  and  $D$  in the real projective plane, the condition whether a polygon with  $n$  sides, which is inscribed in  $D$  and circumscribed about  $C$ , is independent of the starting point of the polygon. In particular, there cannot be such polygons with a different number of sides for a given pair of conics. A remarkable number of different proofs can be found in the literature, ranging from rather elementary proofs for special cases up to proofs using the theory of elliptic curves or measure theory (see [1] for an overview). In addition to proving the statement itself, much work has been done to find criteria for the existence of such polygons for two given conics, the most advanced result given by Arthur Cayley in 1853. The aim of this paper is to look at Poncelet's Theorem for a specific pencil of conics in finite projective coordinate planes  $PG(2, p)$ ,  $p$  an odd prime. We describe a criterion for the existence of Poncelet Polygons in such planes. The most interesting part in this analysis is the connection between the existence of Poncelet Polygons, which can be seen as geometric objects, and the theory of quadratic residues, which is purely number theoretic.

In the first chapter, we introduce some basic notation as well as the most important definitions and results used later on. In particular, we recall some facts about conics in finite projective planes and collineations. In the second chapter, we describe the pencil of conics we are working with as well as its properties. We state a version of Poncelet's Theorem for  $PG(2, p)$  and give a proof for the conic pairs constructed before. Moreover, some crucial properties which help to find conditions about Poncelet Polygons are considered. In the third chapter, we start with a condition for the existence of Poncelet Triangles and reveal a first connection to the theory of quadratic residues. Also, we show how to reduce the problem of finding Poncelet Polygons to those with an odd number of

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sides. A generic example using 5-sided Poncelet Polygons shows how the Euler totient function is involved in Poncelet's Theorem in  $PG(2, p)$ . This enables us to deduce more information about the relation of Poncelet Pairs of conics, which finally leads to the algorithm that allows to find Poncelet Polygons by looking at the conic equations only. In the last chapter, we take a brief look at the Euclidean plane and investigate some parallels to the formulas derived for the finite planes, as for example the half-angle formula, which can henceforth be interpreted in finite planes as well.

## 1 Preliminaries

We start with recollecting the most important definitions and facts about finite projective planes used later on in this paper (see [2]).

**Definition 1.1.** *The triple  $(\mathbb{P}, \mathbb{B}, \mathbb{I})$  with  $\mathbb{I} \subset \mathbb{P} \times \mathbb{B}$  is called projective plane, if the following axioms are satisfied:*

1. *For any two elements  $P, Q \in \mathbb{P}$ ,  $P \neq Q$ , there exists a unique element  $g \in \mathbb{B}$  with  $(P, g) \in \mathbb{I}$  and  $(Q, g) \in \mathbb{I}$ .*
2. *For any two elements  $g, h \in \mathbb{B}$ ,  $g \neq h$ , there exists a unique element  $P \in \mathbb{P}$  with  $(P, g) \in \mathbb{I}$  and  $(P, h) \in \mathbb{I}$ .*
3. *There are four elements  $P_1, \dots, P_4 \in \mathbb{P}$  such that  $\forall g \in \mathbb{B}$  we have  $(P_i, g) \in \mathbb{I}$  and  $(P_j, g) \in \mathbb{I}$  with  $i \neq j$  implies  $(P_k, g) \notin \mathbb{I}$  for  $k \neq i, j$ .*

We are only working with finite projective planes of order  $p$ , for  $p$  an odd prime, i.e.:

**Definition 1.2.** *A finite projective plane  $(\mathbb{P}, \mathbb{B}, \mathbb{I})$  is said to be of order  $p$ , if  $|\mathbb{P}| = |\mathbb{B}| = p^2 + p + 1$ . It will be denoted by  $\mathcal{P}_p$ .*

A particular class of finite projective planes are so-called coordinate planes, which are constructed as follows:

1. The set of points  $\mathbb{P}$  is defined as

$$\mathbb{P} = \{(x, y, z) \in GF(p)^3, (x, y, z) \neq (0, 0, 0)\} / \sim$$

where  $\sim$  is an equivalence relation given by

$$(\lambda x, \lambda y, \lambda z) \sim (x, y, z), \forall \lambda \in GF(p)^* = GF(p) \setminus \{0\}$$

2. Using the same equivalence relation, the set of lines  $\mathbb{B}$  is defined as

$$\mathbb{B} = \{(a, b, c) \in GF(p)^3, (a, b, c) \neq (0, 0, 0)\} / \sim$$

3. The incidence relation  $\mathbb{I}$  is given by the inner product:

$$P = (x, y, z) \in \mathbb{P} \text{ is incident with } g = (a, b, c) \in \mathbb{B} \iff ax + by + cz = 0 \text{ in } GF(p)$$

The finite projective plane of order  $p$  constructed in this way is unique up to isomorphisms and denoted by  $PG(2, p)$ . All points, lines, pairs of lines and conics in  $PG(2, p)$  can be described as solutions of

$$ax^2 + by^2 + cz^2 + dxy + exz + fyz = 0, \quad (1)$$

with  $a, b, c, d, e, f \in GF(p)$ . Another way to look at this quadratic form is to consider  $v^T A v$  for  $v = (x, y, z)$  and

$$A = \begin{pmatrix} a & \frac{d}{2} & \frac{e}{2} \\ \frac{d}{2} & b & \frac{f}{2} \\ \frac{e}{2} & \frac{f}{2} & c \end{pmatrix}$$

Equation (1) then corresponds to a conic if and only if the corresponding matrix  $A$  is regular. If equation (1) corresponds to a singular matrix and is irreducible, the solution is one point only. Otherwise, if the quadratic form splits into two linear factors, it corresponds to one or two lines. In abuse of notation, by a conic we mean the set of points, the quadratic equation as well as the corresponding symmetric matrix, depending on the context. By the Chevalley-Warning Theorem, (1) has at least one non-zero solution  $P = (x, y, z)$ . If the corresponding matrix is regular, by cutting the conic with lines passing through  $P$ , it is easy to see that there are exactly  $p + 1$  points on a conic. As usual, if  $O$  is a given conic,  $P$  a point and  $l$  a line, we call  $l$  a *tangent*, if it has one point in common with  $O$ , a *secant*, if it has two points in common with  $O$  and an *external line* if it misses  $O$ .  $P$  is called *inner point*, if there is no tangent to  $O$  through  $P$ , and *exterior point*, if there are two tangents from  $P$  to  $O$ . It is very convenient to work with the matrix representation of a conic, as can be seen by the following fact:

**Lemma 1.1.** *Let  $O$  be any conic in  $PG(2, p)$  and  $P$  be any point in  $PG(2, p)$ . Then:*

- $P$  is on  $O \Leftrightarrow OP$  is a tangent of  $O$
- $P$  is an exterior point of  $O \Leftrightarrow OP$  is a secant of  $O$
- $P$  is an inner point of  $O \Leftrightarrow OP$  is an external line of  $O$

An important tool are collinear maps of  $PG(2, p)$ :

**Lemma 1.2.** *If  $S$  is a regular  $3 \times 3$  matrix with coefficients in  $GF(p)$ , then  $\phi_S : PG(2, p) \rightarrow PG(2, p)$ ,  $P \mapsto SP$  is bijective and collinear, i.e., a set of collinear points is mapped to a set of collinear points.*

*Remark 1.3.* We may consider the map  $\phi_S$  as a coordinate transformation. Observe, that for three points  $P, Q, R$  in  $PG(2, p)$ , we have:

$$SP, SQ, SR \text{ not collinear} \Leftrightarrow \det(SP, SQ, SR) = \det(S) \det(P, Q, R) \neq 0$$

For a point  $P$  and a line  $l$  in  $PG(2, p)$ , we have:

$$P \in l \Leftrightarrow SP \in (S^T)^{-1}l$$

Moreover, let  $P_1, \dots, P_{p+1}$  be the  $p + 1$  points on a conic  $O$  and  $S$  regular. Then the points  $SP_1, \dots, SP_{p+1}$  lie again on a conic, namely on the conic  $\phi_S(O)$  given by  $(S^{-1})^T OS^{-1}$ . In fact, we have

$$SP \in \phi_S(O) \Leftrightarrow (SP)^T (S^{-1})^T OS^{-1} (SP) = 0 \Leftrightarrow P^T OP = 0 \Leftrightarrow P \in O$$

## 2 A special pencil of conics in $PG(2, p)$

### 2.1 Construction and properties

In all of the following, we only consider conics of the form

$$O_k : x^2 + ky^2 + ckz^2 = 0, \quad 1 \leq k \leq p-1, \quad (2)$$

where the parameter  $c$  is a nonsquare if  $p \equiv 1(4)$ , and a square if  $p \equiv 3(4)$ . To understand the properties of a pair of such conics, we first have a closer look at a specific partition of the plane  $PG(2, p)$ . The idea is to start with the point  $P = (1, 0, 0)$  and the line  $g$  through the points  $(0, 1, 0)$  and  $(0, 0, 1)$ . By looking at all linear combinations of the equations corresponding to  $P$  and  $g$ , we get a partition of the plane consisting of the  $p - 1$  conics  $O_1, \dots, O_{p-1}$  as well as the point  $P$  and the line  $g$ . To see this, look at the following results:

**Lemma 2.1.** *An equation of the point  $P = (1, 0, 0)$  in the plane  $PG(2, p)$  is given by*

$$P : y^2 + cz^2 = 0, \quad (3)$$

*for  $p - c$  not a square in  $GF(p)$ . In particular, for  $p \equiv 1(4)$ , all nonsquares and for  $p \equiv 3(4)$ , all squares in  $GF(p)$  can be used as the parameter  $c$ .*

*Proof.*  $P = (1, 0, 0)$  clearly solves equation (3). As the associated matrix is singular, it describes a point, a line or a pair of lines. It is a point, if the polynomial  $y^2 + cz^2$  is irreducible over  $GF(p)$ . This is the case if and only if  $p - c$  is not a square in  $GF(p)$ . By a well-known theorem in number theory (see [3]), we have:

$$p \equiv 1(4) \Leftrightarrow \exists u, 1 \leq u < p : u^2 \equiv p - 1(p)$$

Using this result, we obtain:

$$c \equiv k^2(p) \Leftrightarrow p - c \equiv p - k^2(p) \Leftrightarrow p - c \equiv (uk)^2(p), \text{ for } u^2 \equiv p - 1(p)$$

So  $c$  is a square if and only if  $p - c$  is a square for  $p \equiv 1(4)$ . Because of this, we can only choose nonsquares for the equation of  $P = (1, 0, 0)$  in such planes. Since  $p - 1$  is not a square in  $GF(p)$ ,  $p \equiv 3(4)$ , we have to choose the squares in that case.  $\square$

In the construction which follows, we start with any point  $P$  and any line  $g$ ,  $P \notin g$ . Since there exists a collineation of  $PG(2, p)$ , which maps any three noncollinear points to any other noncollinear points, we restrict the proofs, without loss of generality, to  $P = (1, 0, 0)$  and  $g$  the line through  $(0, 1, 0)$  and  $(0, 0, 1)$ .

**Lemma 2.2.** *Let  $P$  be a point and  $g$  a line in  $PG(2, p)$ , such that  $P \notin g$ . Then there exist  $p - 1$  conics  $O_1, \dots, O_{p-1}$ , such that the equations corresponding to  $P, g, O_1, \dots, O_{p-1}$  are closed under addition and the zeros of these equations form a partition of the plane  $PG(2, p)$ . Moreover,  $P$  is the unique point in  $PG(2, p)$ , which is an inner point of all conics  $O_1, \dots, O_{p-1}$ .*

*Proof.* Without loss of generality, take  $P = (1, 0, 0)$  and  $g$  the line through  $(0, 1, 0)$  and  $(0, 0, 1)$ . By Lemma 2.1, the corresponding equations are given by

$$g : x^2 = 0 \text{ and } P : y^2 + cz^2 = 0,$$

for  $p - c$  not a square. Considering all nontrivial  $GF(p)$ -linear combinations of  $P$  and  $g$  leads to  $p - 1$  conics, where we define  $O_1 := P + g$  and  $O_k := P + O_{k-1}$ , i.e.,

$$O_k : x^2 + ky^2 + kcz^2 = 0 \text{ for } k = 1, \dots, p - 1$$

Note that these equations are all closed under addition. Moreover, the solutions of these equations are disjoint, since a common solution of any two equations would imply a common solution of  $P$  and  $g$  as well, which contradicts our assumption. Because of this, the solutions of these  $p + 1$  equations give  $p(p + 1) + 1 = p^2 + p + 1$  distinct points, which are indeed all points of  $PG(2, p)$ . Hence, the solutions of the  $p + 1$  equations form a partition of  $PG(2, p)$ .

For the second statement, we first have to show that  $P$  is an inner point of all conics. Note that any point on  $O_k$  has a nonzero  $x$ -coordinate, since otherwise there would be an intersection with the line  $g$ . By Lemma 1.1, for any point  $R = (1, r_2, r_3)$  on  $O_k$  the tangent to  $O_k$  in  $R$  is the line  $(1, kr_2, kcr_3)$ . The point  $P$  is an inner point of  $O_k$ , if it is not incident with any tangent of  $O_k$ . Indeed, we have

$$(1, 0, 0) \cdot (1, kr_2, kcr_3) = 1 \neq 0$$

and hence,  $P$  is not incident with any tangent of  $O_k$  for  $k = 1, \dots, p - 1$ . Therefore, it is an inner point of all conics  $O_1, \dots, O_{p-1}$ . It remains to show that no other point of  $PG(2, p)$  is an inner point of all these conics. Since we can exclude all points lying on some conic  $O_k$ , we just have to deal with the line  $g$ . Note that:

$$g = \{(0, 0, 1), (0, 1, 0), (0, 1, 1), \dots, (0, 1, p - 1)\}$$

Consider the point  $(0, 0, 1)$ . This point is incident with the tangent of  $O_k$  in  $R = (1, r_2, r_3)$ , if  $(1, kr_2, kcr_3) \cdot (0, 0, 1) = kcr_3$  is zero. Since  $k \neq 0$  and  $c \neq 0$ , this is exactly the case for all points of the form  $R = (1, r_2, 0)$ . Because of this,  $(0, 0, 1)$  is not an inner point of those conics containing such points. Since we have a partition of the plane, the existence of such conics is guaranteed. We proceed similarly for the points  $(0, 1, \alpha)$ ,  $\alpha = 0, \dots, p-1$ . Such a point is incident with the tangent of  $O_k$  in  $R = (1, r_2, r_3)$  if  $(1, kr_2, kcr_3) \cdot (0, 1, \alpha) = kr_2 + kcar_3$  is zero. Since  $k \neq 0$ , we need  $r_2 + car_3$  to be zero. The points  $R_\alpha := (1, \alpha, -c^{-1})$  satisfy this condition and for all values of  $\alpha$ ,  $R_\alpha$  is indeed a point on some conic  $O_k$ , since we have a partition of the plane.  $\square$

**Theorem 2.3.** *Let  $P$  be a point and  $g$  a line in  $PG(2, p)$  with  $P \notin g$ . Let  $O_1, \dots, O_{p-1}$  be the  $p-1$  pairwise disjoint conics obtained by adding the equations of  $P$  and  $g$  successively. Then each line through  $P$  is a secant of all  $O_i$ ,  $i$  a square in  $GF(p)$  and an external line of all  $O_j$ ,  $j$  not a square in  $GF(p)$  or vice versa.*

*Proof.* Without loss of generality, take  $P = (1, 0, 0)$  and  $g$  the line through  $(0, 1, 0)$  and  $(0, 0, 1)$ . All  $p+1$  lines through  $P$  are given by

$$s_t : y + tz = 0, \quad t = 0, \dots, p-1 \quad \text{and} \quad s_p : z = 0.$$

Recall, that  $O_k : x^2 + ky^2 + kcz^2 = 0$  for  $k = 1, \dots, p-1$ . We start proving the theorem in planes  $PG(2, p)$ ,  $p \equiv 1(4)$ . Remember that in these planes,  $p-1$  is a square and  $c$  is a nonsquare. We start looking at the line  $s_p$ . This line is a secant of  $O_k$  if the following equation has a solution  $y$  in  $GF(p)$ :

$$1 + ky^2 = 0 \Leftrightarrow y^2 = k^{-1}(p-1)$$

Since  $(p-1)$  is a square, the equation above is solvable if and only if  $k^{-1}$  is a square, which is if and only if  $k$  is a square,  $k \in \{1, \dots, p-1\}$ . Hence,  $z = 0$  is a secant of all  $O_k$ ,  $k$  a square and an external line of the remaining conics  $O_k$ ,  $k$  a nonsquare. For the lines  $s_t$ ,  $t = 0, \dots, p-1$ , we see that  $s_t$  is a secant of  $O_k$ , if the following equation is solvable for  $z$  in  $GF(p)$ :

$$1 + k(t^2 + c)z^2 = 0 \Leftrightarrow z^2 = k^{-1}(t^2 + c)^{-1}(p-1)$$

Note that  $t^2 + c \neq 0$ , since  $p-c$  was chosen to be a nonsquare. The equation is solvable, if  $k(t^2 + c)$  is a square. For this, we find:

$$s_t \text{ is a secant of } O_k \Leftrightarrow k = \begin{cases} \text{square,} & \text{if } (t^2 + c) \text{ square} \\ \text{nonsquare,} & \text{if } (t^2 + c) \text{ nonsquare} \end{cases}$$

Hence,  $s_t$  is either a secant of all  $O_k$ ,  $k$  a square, or all  $O_k$ ,  $k$  not a square, depending on the parameters  $t$  and  $c$ . A similar discussion covers the planes  $PG(2, p)$  for  $p \equiv 3(4)$ .  $\square$

The following property is the main building block for what follows:

**Lemma 2.4.** *None of the conics  $O_1, \dots, O_{p-1}$  have tangents in common.*

*Proof.* Consider the conics  $O_N$  and  $O_S$  given by

$$O_N : x^2 + Ny^2 + cNz^2 = 0 \quad \text{and} \quad O_S : x^2 + Sy^2 + cSz^2 = 0.$$

Let  $(1, N_2, N_3) \in O_N$  and  $(1, S_2, S_3) \in O_S$ , so we have

$$N_2^2 = -N^{-1} - cN_3^2 \quad \text{and} \quad S_2^2 = -S^{-1} - cS_3^2. \quad (4)$$

The corresponding tangents are given by

$$t_{O_N}(1, N_2, N_3) = (1, NN_2, cNN_3) \quad \text{and} \quad t_{O_S}(1, S_2, S_3) = (1, SS_2, cSS_3).$$

Assume these tangents are the same, i.e.  $(1, NN_2, cNN_3) = (1, SS_2, cSS_3)$ . Since they both have the same entry at the first coordinate, we necessarily have

$$NN_2 = SS_2 \quad \text{and} \quad cNN_3 = cSS_3. \quad (5)$$

Combining (4) and (5) gives indeed  $N = S$  which completes the proof.  $\square$

We end this section by showing that the parameter  $c$  can indeed be chosen arbitrarily among all squares or nonsquares, without any changes of incidence relations.

**Lemma 2.5.** *Let  $c_1$  and  $c_2$  be squares in case  $p \equiv 3(4)$  and nonsquares in case  $p \equiv 1(4)$ , respectively. Then the two partitions of  $PG(2, p)$  given by*

$$x^2 = 0, \quad y^2 + c_i z^2 = 0, \quad x^2 + ky^2 + kc_i z^2 = 0, \quad k = 1, \dots, p-1$$

*for  $i = 1, 2$  can be mapped to each other by a collinear transformation of  $PG(2, p)$ .*

*Proof.* In both case,  $c_1 c_2$  is a square, so we can consider the collinear map  $\phi_S$  given by the matrix

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \sqrt{c_2 c_1^{-1}} \end{pmatrix}.$$

If  $O$  is the matrix corresponding to one of the equations defining the partition with parameter  $c_1$ , then  $S^T O S$  is the matrix of the corresponding equation with parameter  $c_2$ .  $\square$

## 2.2 Poncelet's Theorem for conics $O_k$

The main goal in this section is to prove a version of Poncelet's Porism, interpreted in  $PG(2, p)$ . Recall that we are only interested in pairs of conics of the form (2) described in the previous section.

**Definition 2.1.** *Consider a pair of conics  $(O_\alpha, O_\beta)$  given by (2). An  $n$ -sided Poncelet Polygon is a polygon with  $n$  sides and vertices on  $O_\beta$ , such that the sides are tangents of  $O_\alpha$ .*

Since the conics  $O_k$  are all disjoint and have no common tangents, as seen in Lemma 2.4, we are in the particular situation that if we can find one line, which is a tangent to  $O_\alpha$  and a secant of  $O_\beta$ , this leads necessarily to a Poncelet Polygon. The version of Poncelet's Theorem we are going to prove here reads as follows:

**Theorem 2.6.** *Let  $(O_\alpha, O_\beta)$  be any pair of conics in  $PG(2, p)$  with equations of the form*

$$O_k : x^2 + ky^2 + ckz^2, \quad k \in \{\alpha, \beta\}.$$

*If an  $n$ -sided Poncelet Polygon can be constructed starting with a point  $P \in O_\beta$ , then an  $n$ -sided Poncelet Polygon can be constructed starting with any other point  $Q \in O_\beta$  as well.*

*Proof.* We start with the result in planes  $PG(2, p)$ ,  $p \equiv 3(4)$ . By Lemma 2.5, it suffices to consider conics of the form

$$O_k : x^2 + ky^2 + kz^2 = 0. \tag{6}$$

since  $c = 1$  is always a square. The idea is to find a collineation, which does not change the equation of the conics but maps points of the conic to each other. Since the conics in this case can be interpreted as concentric circles, we apply collineations which rotate the  $n$ -sided Poncelet Polygons suitably, i.e. we look at members of the dihedral group of  $p+1$  elements or a subgroup of it. Let  $P_1, P_2, \dots, P_n$  be points on  $O_\beta$  which form an  $n$ -sided Poncelet Polygon with some other conic  $O_\alpha$ . Let  $Q = (1, q_2, q_3)$  be any other point on  $O_\beta$ . Denote  $P_1 = P = (1, p_2, p_3)$ . The goal is to find a collineation  $\tau = \tau_{(P, Q)}$  which maps  $P$  to  $Q$  and does not change the conic equation. For this, look at:

$$\tau = \begin{pmatrix} 1 & 0 & 0 \\ 0 & a & b \\ 0 & b & -a \end{pmatrix}$$

We want  $\tau(P) = Q$  which leads to the two conditions:

$$ap_2 + bp_3 = q_2 \tag{7}$$

$$bp_2 - ap_3 = q_3 \quad (8)$$

We know that  $P$  and  $Q$  are on  $O_\beta$ , which means:

$$p_2^2 + p_3^2 = -k^{-1} \text{ and } q_2^2 + q_3^2 = -k^{-1} \quad (9)$$

In the case  $p_2 \neq 0$  and  $p_3 \neq 0$ , we immediately deduce by combining (7) and (8):

$$a = -kp_2q_2 + kq_3p_3 \text{ and } b = -kq_2p_3 - kq_3p_2$$

Note that  $a$  and  $b$  are elements in  $GF(p)$  with the property  $a^2 + b^2 \equiv 1(p)$ , since using (9) gives:

$$a^2 + b^2 = k^2(q_2^2 + q_3^2)(p_2^2 + p_3^2) = 1$$

Because of this,  $\tau$  does not change the equation of the conic, i.e.  $\tau(O_k) = O_k$ , as can be easily checked. Hence, applying  $\tau_{(P,Q)}$  to all points  $P_i$  for  $i = 1, \dots, n$  yields a new Poncelet Polygon using the point  $Q$ . Similarly, the cases for  $P = (1, 0, p_3)$  and  $P = (1, p_2, 0)$  can be discussed, which completes the proof for planes  $PG(2, p)$ ,  $p \equiv 3(4)$ .

It remains to prove the result in planes  $PG(2, p)$ ,  $p \equiv 1(4)$ . Remember that in these planes, all nonsquares can be taken for the parameter  $c$ . But  $c = 1$  is a square, and we have to set up the transformation differently. Let  $c$  be an arbitrary but fixed nonsquare in  $GF(p)$ . Define

$$GF(p)(\sqrt{c}) := \{a + b\sqrt{c} \mid a, b \in GF(p)\}$$

and consider the usual addition and multiplication, defined by:

$$(x + y\sqrt{c}) + (z + w\sqrt{c}) := (x + z) + (y + w)\sqrt{c}$$

$$(x + y\sqrt{c})(z + w\sqrt{c}) := (xz + ywc) + (yz + xw)\sqrt{c}$$

It is well-known that  $GF(p)(\sqrt{c})$  is indeed a field and isomorphic to  $GF(p^2)$ . Let us go back to the conics  $O_k : k^{-1}x^2 + y^2 + cz^2 = 0$  in  $PG(2, p)$  we started with. The main idea is to embed these conics in  $PG(2, p^2)$ . First note, that  $\varepsilon : PG(2, p) \rightarrow PG(2, p^2)$ ,  $P \mapsto P$  is a natural embedding. Then we choose the collinear transformation  $\phi_s$  in  $PG(2, p^2)$  given by the matrix  $S$  in  $GF(p)(\sqrt{c})$ :

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \sqrt{c} \end{pmatrix}$$

All conics  $O_k$  in  $PG(2, p)$  are mapped by  $\phi_S \circ \varepsilon$  to conics  $S(O_k)$  in  $PG(2, p^2)$ . The equation of  $S(O_k)$  is then given by

$$S(O_k) : k^{-1}x^2 + y^2 + z^2 = 0,$$

since

$$(S^{-1})^T O_k S^{-1} = \begin{pmatrix} k^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

For the tangents of  $O_k$ , we can proceed similarly by considering again the equation of  $S(O_k)$  in  $PG(2, p^2)$ . It is easy to check, that a tangent of  $O_k$  is mapped to a tangent of  $S(O_k)$  by  $\phi_S \circ \varepsilon$ . Now, on the conics  $S(O_k)$  in  $PG(2, p^2)$ , we can operate similarly as before. Let  $P_1, P_2, \dots, P_n$  be points on  $O_\beta$  which form an  $n$ -sided Poncelet Polygon with some other conic  $O_\alpha$ . Let  $Q = (1, q_2, q_3)$  be any other point on  $O_\beta$  and denote  $P_1 = P = (1, p_2, p_3)$ . We look at  $P$  and  $Q$  in  $PG(2, p^2)$ , namely at  $SP = (1, p_2, \sqrt{c}p_3)$  and  $SQ = (1, q_2, \sqrt{c}q_3)$ . The goal is to find a collineation  $\tau = \tau_{(SP, SQ)}$  which maps  $SP$  to  $SQ$  and does not change the conic equation  $S(O_k)$ . Now, we look at:

$$\tau = \begin{pmatrix} 1 & 0 & 0 \\ 0 & a & \sqrt{cb} \\ 0 & \sqrt{cb} & -a \end{pmatrix}$$

The condition  $\tau(SP) = SQ$  leads to:

$$ap_2 + cbp_3 = q_2 \text{ and } bp_2 - ap_3 = q_3$$

Assume again  $p_2 \neq 0$  and  $p_3 \neq 0$ . The other two cases (i.e.  $p_2 = 0$  or  $p_3 = 0$ ) can be carried out similarly.

$$a = -kp_2q_2 + kcq_3p_3 \quad (10)$$

$$b = -kq_2p_3 - kq_3p_2 \quad (11)$$

It can be checked immediately, by using  $P, Q \in O_\beta$ , that:

$$a^2 + cb^2 = 1 \quad (12)$$

Hence,  $\tau$  does not change the equation of  $S(O_k)$  and therefore maps points on  $S(O_k)$  to points on  $S(O_k)$ . We apply  $\tau$  to all points on the embedded Poncelet  $n$ -gon given by  $SP_1, \dots, SP_n$ , which gives a new Poncelet  $n$ -gon for  $S(O_\beta)$  and  $S(O_\alpha)$ . It remains to show that mapping these transformed points back to  $PG(2, p)$  gives points on the original conic  $O_\beta$ , i.e. we have to show that for any  $P_i = (1, y_i, z_i) \in O_\beta$ :

$$S^{-1}(\tau(SP_i)) = \begin{pmatrix} 1 \\ ay_i + cbz_i \\ by_i - az_i \end{pmatrix} \in O_\beta$$

Look at the corresponding equation:

$$1 + k(ay_i + cbz_i)^2 + ck(by_i - az_i)^2 = 1 + ky_i^2(a^2 + cb^2) + kcz_i^2(a^2 + cb^2)$$

Because of (12) and  $P_i \in O_\beta$  this expression indeed equals zero. Hence, we end up with a Poncelet  $n$ -gon for  $O_\beta$  and  $O_\alpha$  starting with the point  $Q$ .  $\square$

## 2.3 Relations for pairs of conics

In this section, we consider the disposition of pairs of conics with regard to the existence of a Poncelet Polygon.

**Definition 2.2.** Let  $O$  and  $O'$  be two conics in  $PG(2, p)$ . We say that  $O$  lies inside  $O'$ , if  $O'$  consists of external points of  $O$  only. Notation:  $O \diamond O'$ .

Note that this relation is not symmetric, since there are conics  $O$  consisting of external points of  $O'$  but the converse is not true. Moreover, we can have the situation that neither  $O$  lies inside  $O'$  nor  $O'$  lies inside  $O$ .

In the following we continue to consider conics described in the previous section, i.e., conics given by the equation  $O_k : x^2 + ky^2 + cz^2 = 0$ .

**Theorem 2.7.** Let  $O_\alpha$  and  $O_\beta$  be conics in  $PG(2, p)$  of the given form,  $p$  an odd prime. If one point  $P \in O_\beta$  is an external point of  $O_\alpha$ , then  $O_\alpha \diamond O_\beta$ . Moreover, we have  $O_\alpha \diamond O_\beta$  if and only if:

$$(-\beta)(\beta - \alpha) = \begin{cases} \text{nonsquare in } GF(p), & p \equiv 1(4) \\ \text{square in } GF(p), & p \equiv 3(4) \end{cases}$$

*Proof.* Let  $O_\alpha$  and  $O_\beta$  be given by:

$$O_\alpha : x^2 + \alpha y^2 + c\alpha z^2 = 0 \text{ and } O_\beta : x^2 + \beta y^2 + c\beta z^2 = 0$$

Remember that all points of  $PG(2, p)$  with a zero  $x$ -coordinate lie on the line  $g : x^2 = 0$ , hence due to the partition not on any conic. A point  $P$  of  $O_\beta$  can therefore be considered as  $P = (1, P_2, P_3)$ . Using the conic equation, we have  $P_2^2 = -\beta^{-1} - cP_3^2$ . By Lemma 1.1, the conic  $O_\alpha$  lies inside  $O_\beta$  if for all such points  $P$ ,  $O_\alpha P$  is a secant of  $O_\alpha$ . So, the property  $O_\alpha \diamond O_\beta$  can be equivalently



expressed as follows:

$$\begin{aligned}
(1, \alpha P_2, c\alpha P_3) \text{ a secant of } O_\alpha &\Leftrightarrow \\
\exists(x, y, z) \in O_\alpha : \alpha^{-1}x + P_2y + cP_3z = 0 &\Leftrightarrow \\
\exists(x, y, z) = (1, \pm\sqrt{-\alpha^{-1} - cz^2}, z) : \alpha^{-1}x + P_2y + cP_3z = 0 &\Leftrightarrow \\
\pm\sqrt{(-\beta^{-1} - cP_3^2)(-\alpha^{-1} - cz^2)} = -\alpha^{-1} - cP_3z &\Leftrightarrow \\
z^2 - 2\alpha^{-1}\beta P_3z + \alpha^{-1}c^{-1} + \alpha^{-1}\beta P_3^2 - \alpha^{-2}\beta c^{-1} = 0 &
\end{aligned}$$

This quadratic equation is solvable for  $z$  iff its discriminant is a square in  $GF(p)$ , i.e., iff

$$\begin{aligned}
\alpha^{-2}\beta^2P_3^2 - \alpha^{-1}c^{-1} - \alpha^{-1}\beta P_3^2 + \alpha^{-2}\beta c^{-1} \text{ square} &\Leftrightarrow \\
\beta^2P_3^2 - \alpha c^{-1} - \alpha\beta P_3^2 + \beta c^{-1} \text{ square} &\Leftrightarrow \\
(\beta P_3^2 + c^{-1})(\beta - \alpha) \text{ square} &\Leftrightarrow \\
(-\beta P_3^2 c^{-1})(\beta - \alpha) \text{ square} &\Leftrightarrow \\
(-\beta c^{-1})(\beta - \alpha) \text{ square} &
\end{aligned}$$

In planes  $PG(2, p)$ ,  $p \equiv 1(4)$ ,  $c$  is chosen to be a nonsquare, hence we need  $(-\beta)(\beta - \alpha)$  to be a nonsquare. In planes  $PG(2, p)$ ,  $p \equiv 3(4)$ , the variable  $c$  is a square, hence we need  $(-\beta)(\beta - \alpha)$  to be a square. Since the above expression is independent of the point  $P$ , it holds for every point on  $O_\beta$  and we are done.  $\square$

With Theorem 2.7, we can now construct chains of nested conics, since we have:

**Corollary 2.8.** *Consider two conics  $O_\alpha$  and  $O_\beta$ . Then:*

$$O_\alpha \diamond O_\beta \Leftrightarrow O_\beta \diamond O_{\beta^2\alpha^{-1}}$$

When calculating the relation  $\diamond$  for every pair in a given plane, it is useful to apply the following result.

**Lemma 2.9.** *Let  $(O_k, O_1)$  be a pair of conics in  $PG(2, p)$ . Then there exists a collinear transformation mapping  $(O_k, O_1)$  to  $(O_{\beta k}, O_\beta)$ , for all  $\beta \in GF(p) \setminus \{0\}$ . In particular,  $O_k \diamond O_1$  implies  $O_{\beta k} \diamond O_\beta$ .*

*Proof.* We have to find a collinear transformation  $\phi_S$  with:

$$\phi_S(O_k) = O_{\beta k}$$

Let us start with the case of  $\beta$  being a square in  $GF(p)$ . In this case, we are allowed to consider the square root of  $\beta$ , hence the following regular matrix  $S$  works, which can be checked immediately:

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sqrt{\beta}^{-1} & 0 \\ 0 & 0 & \sqrt{\beta}^{-1} \end{pmatrix}$$

For  $\beta$  not a square, we have to distinguish between  $p \equiv 3(4)$  and  $p \equiv 1(4)$ . In the first case, we can take any square for the parameter  $c$  in the conic equation, as seen in Lemma 2.7. Since the relation of the conics does not depend on  $c$  by Lemma 2.5, we can take  $c = 1$ . Now, choose any nonzero square  $s$  in  $GF(p)$  such that  $\beta - s$  is a square as well. Note that we can always find such an  $s$ , because  $\beta - s$  runs for  $s$  a square through exactly  $\frac{p-1}{2}$  values, since there are  $\frac{p-1}{2}$  squares in  $GF(p) \setminus \{0\}$ . Then  $\beta - s \neq \beta$ , since we take  $s$  not to be zero. Hence, not all of those  $\frac{p-1}{2}$  values can be nonsquares. Moreover,  $\beta - s \neq 0$  as  $s$  is a square and  $\beta$  a nonsquare. Hence, there is a

nonzero square  $\beta - s$ . Therefore, the square roots of  $s$  and  $\beta - s$  are well-defined. Now, look at the following matrix:

$$S^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sqrt{s} & -\sqrt{\beta-s} \\ 0 & \sqrt{\beta-s} & \sqrt{s} \end{pmatrix}$$

We have  $\det(S^{-1}) = \beta \neq 0$  and moreover, it holds:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \sqrt{s} & \sqrt{\beta-s} \\ 0 & -\sqrt{\beta-s} & \sqrt{s} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sqrt{s} & -\sqrt{\beta-s} \\ 0 & \sqrt{\beta-s} & \sqrt{s} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & k\beta & 0 \\ 0 & 0 & k\beta \end{pmatrix}$$

This gives the valid coordinate transformation we were looking for. In planes  $PG(2, p)$ ,  $p \equiv 1(4)$ , we can take any nonsquare for our parameter  $c$ . By Lemma 2.5, the relation of the conics does not depend on  $c$ , hence we choose a  $c$ , such that  $\beta - c$  is a nonzero square. To ensure the existence can be proceeded as before. For such a  $c$ , look at the matrix:

$$S^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sqrt{\beta-c} & c \\ 0 & 1 & -\sqrt{\beta-c} \end{pmatrix}$$

We have  $\det(S^{-1}) = -\beta \neq 0$  and it can be checked that this is the collinear transformation we are looking for.  $\square$

*Example 2.10.* We want to investigate the relation  $\diamond$  in  $PG(2, 7)$ . Here,  $p \equiv 3(4)$ , hence  $O_\alpha \diamond O_\beta$  if and only if  $(-\beta)(\beta - \alpha)$  is a square in  $GF(7)$ , i.e. equals 1, 2 or 4. By looking at  $\beta = 1$  and shifting the result using Lemma 2.9, we obtain the table of relations for the whole plane  $PG(2, 7)$ .

	$O_1$	$O_2$	$O_3$	$O_4$	$O_5$	$O_6$
$O_1$			$\diamond$	$\diamond$	$\diamond$	
$O_2$	$\diamond$		$\diamond$			$\diamond$
$O_3$	$\diamond$	$\diamond$			$\diamond$	
$O_4$		$\diamond$			$\diamond$	$\diamond$
$O_5$	$\diamond$			$\diamond$		$\diamond$
$O_6$		$\diamond$	$\diamond$	$\diamond$		

Using Corollary 2.8, we detect the following closed chains of conics  $O_\alpha \rightarrow O_\beta \rightarrow O_{\beta^2\alpha^{-1}} \rightarrow \dots$

$$\begin{aligned} O_1 &\rightarrow O_3 \rightarrow O_2 \rightarrow O_6 \rightarrow O_4 \rightarrow O_5 \rightarrow O_1 \\ O_1 &\rightarrow O_4 \rightarrow O_2 \rightarrow O_1 \\ O_3 &\rightarrow O_5 \rightarrow O_6 \rightarrow O_3 \end{aligned}$$

Note that starting with two squares  $\alpha$  and  $\beta$  results in a chain of conics with just squares as indices. Similarly, starting with two nonsquares as indices results in a chain of conics with only nonsquares as indices. This shows a connection of this property with cyclotomic subsets, defined by

$$C_i^p(q) := \{qi^n(p), 1 \leq n \leq p-1\}.$$

In  $GF(7)$ , we have for example  $C_3^7(1) = \{3, 2, 6, 4, 5, 1\}$ ,  $C_2^7(1) = \{2, 4, 1\}$  and  $C_2^7(3) = \{6, 5, 3\}$ .

Since exactly half of all nonzero elements in  $GF(p)$  are squares, another immediate result is:

**Corollary 2.11.** *For every conic  $O_\beta$  in  $PG(2, p)$ , there are  $\frac{p-1}{2}$  conics  $O_\alpha$  such that  $O_\alpha \diamond O_\beta$ .*

Next, we have a closer look at the relations of the points on  $O_\alpha$  and  $O_\beta$ .

**Lemma 2.12.** *Let  $P = (1, P_2, P_3)$  be any point on  $O_\beta$  and  $O_\alpha \diamond O_\beta$ . Then, for the contact points  $A_1 = (1, y_1, z_1)$  and  $A_2 = (1, y_2, z_2)$  on  $O_\alpha$  of the tangents through  $P$  we have*

$$z_{1,2} = \alpha^{-1}\beta P_3 \pm P_2 \sqrt{\alpha^{-2}(-c^{-1}\beta)(\beta - \alpha)}$$

and

$$y_{1,2} = \begin{cases} P_2^{-1}(-\alpha^{-1} - cP_3z_{1,2}), & \text{if } P_2 \neq 0 \\ \pm \sqrt{-\alpha^{-1} - cz_{1,2}^2}, & \text{if } P_2 = 0. \end{cases}$$

*Proof.* To see this, we just have to solve the quadratic equation derived in Theorem 2.7. Since  $O_\alpha \diamond O_\beta$ , we indeed get two solutions.  $\square$

**Lemma 2.13.** *Let  $P$  and  $Q$  be two points on  $O_\beta$  such that the line connecting  $P$  and  $Q$  is a tangent of  $O_\alpha$  in the point  $A$ . Then*

$$P + Q = A.$$

*Proof.* Let  $P = (1, P_2, P_3)$  and  $Q = (1, Q_2, Q_3)$  be two points on  $O_\beta$ , so we have:

$$1 + \beta P_2^2 + c\beta P_3^2 = 0 \text{ and } 1 + \beta Q_2^2 + c\beta Q_3^2 = 0$$

There are  $p + 1$  points on the line through  $P$  and  $Q$ , namely

$$\overline{PQ} = \{P, Q, P + Q, P + 2Q, \dots, P + (p - 1)Q\}.$$

Note that  $P + (p - 1)Q$  has a zero  $x$ -coordinate and hence gives the intersection with the line  $g : x^2 = 0$ , which is clearly not a point on  $O_\alpha$ . So we know that

$$\exists! k \in \{1, 2, \dots, p - 2\} : P + kQ = (1 + k, P_2 + kQ_2, P_3 + kQ_3) = A \in O_\alpha \quad (13)$$

We need:

$$\begin{aligned} (k + 1, P_2 + kQ_2, P_3 + kQ_3) \in O_\alpha &\Leftrightarrow \\ (k + 1)^2 + \alpha(P_2 + kQ_2)^2 + c\alpha(P_3 + kQ_3)^2 = 0 &\Leftrightarrow \\ k^2 + \frac{2 + 2\alpha(P_2Q_2 + cP_3Q_3)}{1 - \alpha\beta^{-1}}k + 1 = 0 \end{aligned}$$

Note that  $1 - \alpha\beta^{-1} \neq 0$ , since otherwise  $\alpha = \beta$ . Solving for  $k$  yields

$$k = -\frac{1 + \alpha(P_2Q_2 + cP_3Q_3)}{1 - \alpha\beta^{-1}} \pm \sqrt{\left(\frac{1 + \alpha(P_2Q_2 + cP_3Q_3)}{1 - \alpha\beta^{-1}}\right)^2 - 1}.$$

Note that by (13), there can only be one solution to our problem, since we are looking for a tangent of  $O_\alpha$ . Because of this, the radicand has to be zero, which is if and only if  $\left(\frac{1 + \alpha(P_2Q_2 + cP_3Q_3)}{1 - \alpha\beta^{-1}}\right)^2 = 1$ . Hence  $k = 1$  or  $k = p - 1$ . Since we already excluded  $p - 1$ , we obtain  $k = 1$ , which indeed gives  $P + Q = A$ .  $\square$

**Corollary 2.14.** *Let  $P, Q \in O_\beta$  such that  $(1, 0, 0) \notin \overline{PQ}$ . Then there exists an  $\alpha \in \{1, 2, \dots, p - 1\}$ ,  $\alpha \neq \beta$ , such that  $\overline{PQ}$  is a tangent of  $O_\alpha$ . The contact point is  $P + Q$ .*

*Proof.* With  $P = (1, P_2, P_3), Q = (1, Q_2, Q_3)$ , we have  $P + Q = (2, P_2 + Q_2, P_3 + Q_3)$ . As the characteristic of  $GF(p)$  is odd,  $P + Q$  is not in  $g$ , where  $g$  is the unique line through  $(0, 1, 0)$  and  $(0, 0, 1)$ . Since we have a partition of the plane  $PG(2, p)$ ,  $P + Q$  must be a point on a conic  $O_\alpha$ . We have to exclude the possibility of  $\overline{PQ}$  being a secant of  $O_\alpha$ . For this, note that there are  $p + 1$  points on  $\overline{PQ}$ , where  $P, Q \in O_\beta$  and  $P + (p - 1)Q \in g$ . Hence, there are  $p - 2$  points left, which is an odd number. All the other  $p - 2$  points must lie on conics and there are at most two points on the same conic. Since  $p - 2$  is odd and by Lemma 2.4, there is exactly one conic with  $\overline{PQ}$  as a tangent. By Lemma 2.13, we are done.  $\square$

In the following results, an  $n$ -sided Poncelet Polygon for  $O_\alpha \diamond O_\beta$  with vertices  $B_i$  on  $O_\beta$  and contact points  $A_i$  on  $O_\alpha$  is denoted by

$$B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} B_3 \xrightarrow{A_3} \dots \xrightarrow{A_{n-1}} B_n \xrightarrow{A_n} B_1,$$

where  $B_i \xrightarrow{A_i} B_{i+1}$  means that the line connecting  $B_i$  and  $B_{i+1}$  is the tangent of  $O_\alpha$  in the point  $A_i$ . By Lemma 2.13, the following relations are immediate:

$$B_1 + B_2 = A_1, B_2 + B_3 = A_2, \dots, B_{n-1} + B_n = A_{n-1}, B_n + B_1 = A_n \quad (14)$$

Note that by combining Lemma 2.12 and Lemma 2.13, we are now able to calculate a Poncelet Polygon by starting in a point on  $O_\beta$ . Before we analyze Poncelet Polygons for different numbers of sides, we need some more properties of the points on  $O_k$  and their relations.

**Lemma 2.15.** *The conics  $O_\alpha$  in  $PG(2, p)$ ,  $p \equiv 3(4)$ , consist of the  $p + 1$  points*

$$\left\{ \begin{pmatrix} 1 \\ y_1 \\ z_1 \end{pmatrix}, \begin{pmatrix} 1 \\ -y_1 \\ z_1 \end{pmatrix}, \begin{pmatrix} 1 \\ y_1 \\ -z_1 \end{pmatrix}, \begin{pmatrix} 1 \\ -y_1 \\ -z_1 \end{pmatrix}, \dots, \begin{pmatrix} 1 \\ y_{\frac{p-1}{4}} \\ z_{\frac{p-1}{4}} \end{pmatrix}, \begin{pmatrix} 1 \\ -y_{\frac{p-1}{4}} \\ z_{\frac{p-1}{4}} \end{pmatrix}, \begin{pmatrix} 1 \\ y_{\frac{p-1}{4}} \\ -z_{\frac{p-1}{4}} \end{pmatrix}, \begin{pmatrix} 1 \\ -y_{\frac{p-1}{4}} \\ -z_{\frac{p-1}{4}} \end{pmatrix} \right\}$$

*if  $\alpha$  is a square, and otherwise*

$$\left\{ \begin{pmatrix} 1 \\ y_1 \\ z_1 \end{pmatrix}, \begin{pmatrix} 1 \\ -y_1 \\ z_1 \end{pmatrix}, \begin{pmatrix} 1 \\ y_1 \\ -z_1 \end{pmatrix}, \begin{pmatrix} 1 \\ -y_1 \\ -z_1 \end{pmatrix}, \dots, \begin{pmatrix} 1 \\ y_{\frac{p-3}{4}} \\ -z_{\frac{p-3}{4}} \end{pmatrix}, \begin{pmatrix} 1 \\ -y_{\frac{p-3}{4}} \\ -z_{\frac{p-3}{4}} \end{pmatrix}, \begin{pmatrix} 1 \\ y \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -y \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ z \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -z \end{pmatrix} \right\}$$

*Proof.* Of course, for  $y \neq 0$ ,  $z \neq 0$ , we have that  $(1, y, z) \in O_\alpha$  implies that  $(1, -y, z) \in O_\alpha$ ,  $(1, y, -z) \in O_\alpha$  and  $(1, -y, -z) \in O_\alpha$ . So we just have to check whether  $(1, 0, z)$  and  $(1, y, 0)$  are on  $O_\alpha$  as well. We have:

$$\begin{aligned} (1, 0, z) \in O_\alpha &\Leftrightarrow 1 + cz^2 = 0 \Leftrightarrow z^2 = -\alpha^{-1}c^{-1} \\ (1, y, 0) \in O_\alpha &\Leftrightarrow 1 + \alpha y^2 = 0 \Leftrightarrow y^2 = -\alpha^{-1} \end{aligned}$$

As  $p \equiv 3(4)$ ,  $c$  is a square in  $GF(p)$  and  $p - 1$  is not. Hence these points lie on  $O_\alpha$  if and only if  $\alpha$  is not a square.  $\square$

**Lemma 2.16.** *The conics  $O_\alpha$  in  $PG(2, p)$ ,  $p \equiv 1(4)$ , consist of the  $p + 1$  points*

$$\left\{ \begin{pmatrix} 1 \\ y_1 \\ z_1 \end{pmatrix}, \begin{pmatrix} 1 \\ -y_1 \\ z_1 \end{pmatrix}, \begin{pmatrix} 1 \\ y_1 \\ -z_1 \end{pmatrix}, \begin{pmatrix} 1 \\ -y_1 \\ -z_1 \end{pmatrix}, \dots, \begin{pmatrix} 1 \\ y_{\frac{p-1}{4}} \\ z_{\frac{p-1}{4}} \end{pmatrix}, \begin{pmatrix} 1 \\ -y_{\frac{p-1}{4}} \\ z_{\frac{p-1}{4}} \end{pmatrix}, \begin{pmatrix} 1 \\ y_{\frac{p-1}{4}} \\ -z_{\frac{p-1}{4}} \end{pmatrix}, \begin{pmatrix} 1 \\ -y_{\frac{p-1}{4}} \\ -z_{\frac{p-1}{4}} \end{pmatrix}, \begin{pmatrix} 1 \\ y \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -y \\ 0 \end{pmatrix} \right\}$$

*if  $\alpha$  is a square, and otherwise*

$$\left\{ \begin{pmatrix} 1 \\ y_1 \\ z_1 \end{pmatrix}, \begin{pmatrix} 1 \\ -y_1 \\ z_1 \end{pmatrix}, \begin{pmatrix} 1 \\ y_1 \\ -z_1 \end{pmatrix}, \begin{pmatrix} 1 \\ -y_1 \\ -z_1 \end{pmatrix}, \dots, \begin{pmatrix} 1 \\ y_{\frac{p-1}{4}} \\ z_{\frac{p-1}{4}} \end{pmatrix}, \begin{pmatrix} 1 \\ -y_{\frac{p-1}{4}} \\ z_{\frac{p-1}{4}} \end{pmatrix}, \begin{pmatrix} 1 \\ y_{\frac{p-1}{4}} \\ -z_{\frac{p-1}{4}} \end{pmatrix}, \begin{pmatrix} 1 \\ -y_{\frac{p-1}{4}} \\ -z_{\frac{p-1}{4}} \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ z \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -z \end{pmatrix} \right\}$$

*Proof.* Again, for  $y \neq 0$ ,  $z \neq 0$ , we have that  $(1, y, z) \in O_\alpha$  if and only if  $(1, -y, z) \in O_\alpha$ ,  $(1, y, -z) \in O_\alpha$  and  $(1, -y, -z) \in O_\alpha$ . So we just have to check whether  $(1, 0, z)$  and  $(1, y, 0)$  are on the conics as well. Note that  $c$  is not a square and  $p - 1$  is a square in  $GF(p)$ ,  $p \equiv 1(4)$ , so similarly to the result before, we get:

$$\begin{aligned} (1, 0, z) \in O_\alpha &\Leftrightarrow 1 + cz^2 = 0 \Leftrightarrow z^2 = -\alpha^{-1}c^{-1} \Leftrightarrow \alpha \text{ not a square} \\ (1, y, 0) \in O_\alpha &\Leftrightarrow 1 + \alpha y^2 = 0 \Leftrightarrow y^2 = -\alpha^{-1} \Leftrightarrow \alpha \text{ a square} \end{aligned} \quad \square$$

**Corollary 2.17.** *The sum of all points on the conic  $O_\alpha$  is  $(1, 0, 0)$ .*

*Proof.* This can be seen by checking all possible cases above.  $\square$

**Lemma 2.18.** *Let  $B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} B_3 \xrightarrow{A_3} \dots \xrightarrow{A_{n-1}} B_n \xrightarrow{A_n} B_1$  be an  $n$ -sided Poncelet Polygon. Then*

$$B_1 + B_2 + \dots + B_n = (1, 0, 0) = A_1 + A_2 + \dots + A_n.$$

*Moreover, for  $n$  even, we have*

$$A_1 + A_3 + \dots + A_{n-1} = (1, 0, 0) = A_2 + A_4 + \dots + A_n.$$

*Proof.* By adding the equations in (14) in different ways, we obtain three conditions for  $n$  even:

$$B_1 + B_2 + \dots + B_n = A_1 + A_3 + \dots + A_{n-1} \quad (15)$$

$$B_1 + B_2 + \dots + B_n = A_2 + A_4 + \dots + A_n \quad (16)$$

$$2(B_1 + B_2 + \dots + B_n) = A_1 + A_2 + \dots + A_{n-1} + A_n \quad (17)$$

Combining (15) and (17) gives  $A_2 + A_4 + \dots + A_n = (1, 0, 0)$ , and combining (16) and (17) gives  $A_1 + A_3 + \dots + A_{n-1} = (1, 0, 0)$ . Hence,  $B_1 + B_2 + \dots + B_n = A_1 + A_2 + A_3 + \dots + A_n = (1, 0, 0)$ . Note that the point  $(1, 0, 0)$  operates here as a neutral element concerning addition of points, since  $(0, 0, 0)$  is not a point in  $PG(2, p)$ . The case  $n$  odd is similar.  $\square$

**Lemma 2.19.** *The lines joining opposite vertices  $B_i$  and  $B_{n+i}$  of a  $2n$ -sided Poncelet Polygon meet in  $(1, 0, 0)$ . Moreover  $B_i + B_{n+i} = (1, 0, 0)$ .*

*Proof.* Again, we can use the relation  $B_i + B_{i+1} = A_i$  for the  $2n$ -sided Poncelet Polygon given by points  $B_i \in O_\beta$  and  $A_i \in O_\alpha$ . We have:

$$B_1 + B_{n+1} = A_{2n} - B_{2n} + A_n - B_n$$

$$B_1 + B_{n+1} = A_1 - B_2 + A_{n+1} - B_{n+2}$$

Adding these equations gives

$$2(B_1 + B_{n+1}) = B_1 + B_{n+1} = A_1 - B_2 + A_{n+1} - B_{n+2} + A_{2n} - B_{2n} + A_n - B_n.$$

Taking all  $B_i$  to the left and all  $A_i$  to the right side gives:

$$B_1 + B_2 + B_n + B_{n+1} + B_{n+2} + B_{2n} = A_1 + A_n + A_{n+1} + A_{2n}$$

To apply Lemma 2.18, we add the remaining  $B_i$  to obtain  $(1, 0, 0)$  on the left side:

$$(1, 0, 0) = (A_1 + A_n + A_{n+1} + A_{2n}) + (B_3 + \dots + B_{n-1} + B_{n+3} + \dots + B_{2n-1})$$

Rewriting the  $A_i$  in terms of  $B_i$  gives

$$(1, 0, 0) = (1, 0, 0) + B_1 + B_{n+1}.$$

Hence we get

$$(1, 0, 0) = B_1 + B_{n+1}.$$

Similarly can be proceeded for all remaining  $B_i + B_{n+i}$ ,  $1 < i \leq n$ .  $\square$

Note that Lemma 2.19 can be seen as a generalization of Brianchon's Theorem [4].

### 3 A Poncelet Criterion

#### 3.1 Poncelet Coefficients

Here is a first result concerning the existence of  $n$ -sided Poncelet Polygons.

**Lemma 3.1.** *Let  $O_\alpha \diamond O_\beta$  be two conics in  $PG(2, p)$  which carry a Poncelet Triangle. Then  $4\beta = \alpha$  in  $GF(p)$ .*

*Proof.* Let  $B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} B_3 \xrightarrow{A_3} B_1$  be a Poncelet Triangle. By Lemma 2.13, we therefore have

$$B_1 + B_2 = 2A_1, \quad B_2 + B_3 = 2A_2, \quad B_3 + B_1 = 2A_3.$$

Moreover, by Lemma 2.18, we have  $B_1 + B_2 + B_3 = (1, 0, 0)$ . This gives the following relations:

$$B_1 + 2A_2 = (1, 0, 0), \quad B_2 + 2A_3 = (1, 0, 0), \quad B_3 + 2A_1 = (1, 0, 0) \quad (18)$$

As all lines are given by linear combinations of two points, the conditions in (18) translate to

$$(1, 0, 0) \in \overline{B_1 A_2}, \quad (1, 0, 0) \in \overline{B_2 A_3}, \quad (1, 0, 0) \in \overline{B_3 A_1}.$$

Since there are no tangents through the point  $(1, 0, 0)$ , as seen in Lemma 2.2, these lines have to be secants of  $O_\alpha$  and  $O_\beta$ . With Theorem 2.3, we know that  $\alpha$  and  $\beta$  are either both squares or both nonsquares. To find the remaining intersection points of  $\overline{B_1 A_2}$ ,  $\overline{B_2 A_3}$  and  $\overline{B_3 A_1}$  with  $O_\alpha$  and  $O_\beta$ , consider the points  $\tilde{A}_i$  and  $\tilde{B}_i$ , where  $\tilde{P} := (x, -y, -z)$  for a point  $P = (x, y, z)$ . Since  $(1, 0, 0) \in \overline{B_i \tilde{B}_i}$  and  $(1, 0, 0) \in \overline{A_i \tilde{A}_i}$ , for  $i = 1, 2, 3$ , these are exactly the intersection points we are looking for. Note that this construction yields another Poncelet Triangle: The second triangle is  $\tilde{B}_1 \xrightarrow{\tilde{A}_1} \tilde{B}_2 \xrightarrow{\tilde{A}_2} \tilde{B}_3 \xrightarrow{\tilde{A}_3} \tilde{B}_1$ , as visualized in Figure 3.1. Now, look at  $B_1 = (1, y_1, z_1)$ . The secant

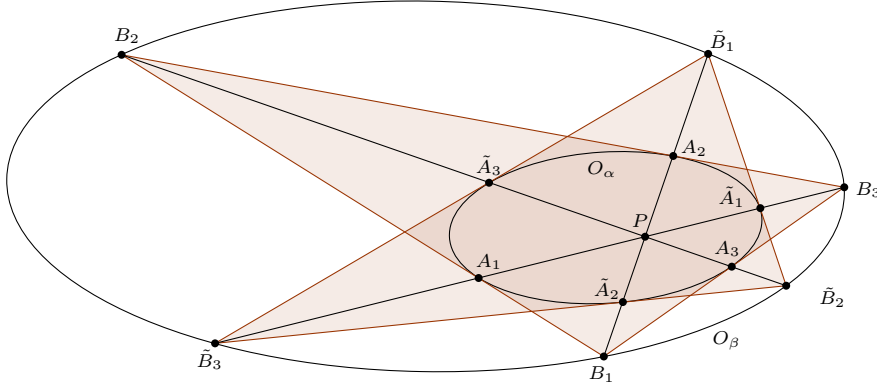


Figure 1: The triangle  $B_1, B_2, B_3$  induces another triangle  $\tilde{B}_1, \tilde{B}_2, \tilde{B}_3$  via  $P = (1, 0, 0)$ .

of  $O_\beta$  through  $B_1$  and  $\tilde{B}_1$  is given by

$$s_1 : z_1 y - y_1 z = 0.$$

In the case  $z_1 \neq 0$ , we get the relation  $y = \frac{y_1}{z_1} z$ . Intersecting this line with the conic  $O_\alpha$  gives:

$$1 + \alpha \left( \frac{y_1}{z_1} z \right)^2 + \alpha z^2 = 0 \Leftrightarrow z^2 = \frac{-z_1^2}{\alpha y_1^2 + \alpha z_1^2}$$

Using  $B_1 \in O_\beta$  gives

$$z^2 = \alpha^{-1} \beta z_1^2.$$

With this, we can calculate the following two intersection points for  $O_\alpha$ :

$$A_2 = (1, y_1 \sqrt{\alpha^{-1}\beta}, z_1 \sqrt{\alpha^{-1}\beta}), \tilde{A}_2 = (1, -y_1 \sqrt{\alpha^{-1}\beta}, -z_1 \sqrt{\alpha^{-1}\beta})$$

Using (18), we obtain the condition

$$(1 + 2\sqrt{\alpha^{-1}\beta})z_1 = 0$$

Since we are in the case  $z_1 \neq 0$ , it follows  $1 + 2\sqrt{\alpha^{-1}\beta} = 0$ , which implies  $\alpha = 4\beta$ . In the case  $z_1 = 0$ , we directly deduce  $z = 0$  for the secant through  $B_1$  and  $\tilde{B}_1$ . Intersecting with  $O_\alpha$  gives the two points

$$A_2 = (1, \sqrt{-\alpha^{-1}}, 0), \tilde{A}_2 = (1, -\sqrt{-\alpha^{-1}}, 0)$$

Applying (18), we get the condition  $y_1 \pm 2\sqrt{-\alpha^{-1}} = 0$  and using  $B_1 \in O_\beta$  yields again  $4\beta = \alpha$ .  $\square$

*Remark 3.2.* Recall that for  $O_\alpha \diamond O_\beta$ , we have to check whether or not  $(-\beta)(\beta - \alpha)$  is a square. Hence, in the case  $4\beta = \alpha$ , we have to check whether or not  $3\beta^2$  is a square, which is the same as checking, whether or not 3 is a square. Compared to results from number theory (see [3]), we indeed have the following conditions for 3 being a square :

- For  $p \equiv 1(4)$ , we have  $3|(p+1) \Leftrightarrow 3$  nonsquare
- For  $p \equiv 3(4)$ , we have  $3|(p+1) \Leftrightarrow 3$  square

This gives already a necessary condition for the existence of Poncelet Triangles for pairs  $(O_\alpha, O_\beta)$  in  $PG(2, p)$ . By Poncelet's Theorem for such pairs, as seen in Theorem 2.6, the existence of a Poncelet Triangle implies  $3|(p+1)$ , as there are  $p+1$  points on the conic  $O_\beta$ . This is exactly the condition given by number theoretic results as well.

Using arguments as above, one easily checks the following result.

**Lemma 3.3.** *Let  $O_\alpha \diamond O_\beta$  be two conics in  $PG(2, p)$ , such that a 4-sided Poncelet Polygon can be constructed. Then  $2\beta = \alpha$  in  $GF(p)$ .*

The main goal is to find such a relation for all possible  $n$ -sided Poncelet Polygons. For this, we first investigate, which Poncelet  $n$ -gons occur in a given plane  $PG(2, p)$ . Note that this can be done just by applying Poncelet's Theorem and the Euler divisor sum formula, since we are dealing with a very special family, as the following results show:

**Lemma 3.4.** *For a given conic  $O_\beta$  in  $PG(2, p)$  and every  $n|(p+1)$ , there are exactly  $\frac{\phi(n)}{2}$  conics  $O_\alpha$ , such that  $O_\alpha \diamond O_\beta$  carries a Poncelet  $n$ -gon.*

*Proof.* By Lemma 2.11, there are exactly  $\frac{p-1}{2}$  conics  $O_\alpha$ , such that  $O_\alpha \diamond O_\beta$ . Moreover, we know that once  $O_\alpha \diamond O_\beta$ , starting with any point of  $O_\beta$  leads to a Poncelet Polygon. Because of Theorem 2.6, the length of this Poncelet Polygon has to divide  $p+1$ , i.e. the number of points on  $O_\beta$ . Recall now Euler's divisor sum formula for the totient function (see [3]):

$$\sum_{n|m} \phi(n) = m$$

Applied to the points of the conic, we have:

$$\sum_{n|(p+1)} \phi(n) = p+1 \Leftrightarrow \sum_{n|(p+1), n \geq 3} \phi(n) = p-1 \Leftrightarrow \sum_{n|(p+1), n \geq 3} \frac{\phi(n)}{2} = \frac{p-1}{2}$$

Hence, there have to be exactly  $\frac{\phi(n)}{2}$  conics  $O_\alpha$  such that  $O_\alpha \diamond O_\beta$  carries a Poncelet  $n$ -gon for every divisor  $n$  of  $p+1$ .  $\square$

We are interested in a criterion, which ensures the existence of an  $n$ -sided Poncelet Polygon for two related conics  $O_\alpha \diamond O_\beta$ , such as  $4\beta = \alpha$  for triangles. The next result reduces this problem of possibly finding such a relation for all  $n$ -sided Poncelet Polygons to those with  $n$  odd.

**Lemma 3.5.** *Let  $(O_{\beta k}, O_\beta)$  be a pair of conics in  $PG(2, p)$ , which carries an  $n$ -sided Poncelet Polygon for  $k$  a square in  $GF(p)$ . Then  $(O_{\beta \tilde{k}}, O_\beta)$  carries a  $2n$ -sided Poncelet Polygon, where*

$$\tilde{k} = \frac{2}{1 - \frac{1}{\sqrt{k}}}$$

where only those roots are taken such that  $\tilde{k} \neq k$ .

*Proof.* Let  $O_\alpha \diamond O_\beta$  be some pair of conics which carries a  $2n$ -sided Poncelet Polygon. To calculate the relation between  $\alpha$  and  $\beta$ , we use that  $B_i + B_{i+1} = A_i$  for two consecutive vertices of the polygon, as seen in Lemma 2.13. Hence:

$$B_1 + B_2 \in O_\alpha \text{ i.e. } (2, y_1 + y_2, z_1 + z_2) \in O_\alpha$$

This gives the following equation to solve:

$$4 + \alpha(y_1 + y_2)^2 + c\alpha(z_1 + z_2)^2 = 0 \Rightarrow \alpha = \frac{-4}{(y_1 + y_2)^2 + c(z_1 + z_2)^2}$$

Since  $B_1 \in O_\beta$  and  $B_2 \in O_\beta$ , we know that  $y_i^2 + cz_i^2 = -\beta^{-1}$  for  $i = 1, 2$  and we obtain:

$$\alpha = \frac{2\beta}{1 - \beta(y_1 y_2 + cz_1 z_2)}$$

The claim is  $\alpha = \beta \tilde{k}$ , hence we have to show:

$$\begin{aligned} \frac{2\beta}{1 - \beta(y_1 y_2 + cz_1 z_2)} &= \frac{2\beta}{1 - \frac{1}{\sqrt{k}}} \Leftrightarrow \\ 1 - \beta(y_1 y_2 + cz_1 z_2) &= 1 - \frac{1}{\sqrt{k}} \Leftrightarrow \\ \frac{1}{\sqrt{k}} + \beta(-y_1)y_2 + c\beta(-z_1)z_2 &= 0 \end{aligned}$$

The expression above can be interpreted as an inner product. Hence, we may reformulate the condition as an incidence relation:

$$\left(\frac{1}{\sqrt{k}}, -y_1, -z_1\right) \cdot (1, \beta y_2, c\beta z_2) = 0 \Leftrightarrow \left(\frac{1}{\sqrt{k}}, -y_1, -z_1\right) \in t_\beta(B_2)$$

This can be done for all pairs of points  $B_i, B_{i+1} \in O_\beta$ . We get the following conditions

$$\left(\frac{1}{\sqrt{k}}, -y_{2\ell-1}, -z_{2\ell-1}\right), \left(\frac{1}{\sqrt{k}}, -y_{2\ell+1}, -z_{2\ell+1}\right) \in t_\beta(B_{2\ell}) \text{ for } \ell = 1, \dots, n,$$

where indices are taken cyclically. Exactly  $n$  tangents of the conic  $O_\beta$  are involved. The conditions above are equivalent to showing that the  $n$  intersection points are on some conic  $O_\gamma$  and form an  $n$ -sided Poncelet Polygon with  $O_\beta$ . Observe that, by Lemma 2.19,  $B_{i+n} = \tilde{B}_i$ , and hence  $\left(\frac{1}{\sqrt{k}}, -y_{i+n}, -z_{i+n}\right) = \left(\frac{1}{\sqrt{k}}, y_i, z_i\right)$ . Therefore, we have to verify that

$$\left(\frac{1}{\sqrt{k}}, \pm y_i, \pm z_i\right) \in O_\gamma, \quad i = 1, \dots, n, \quad \beta = \gamma k$$

We directly obtain the following equation for  $\gamma$ :

$$O_\gamma : \frac{x^2}{k} + \gamma y^2 + c\gamma z^2 = 0$$

Since all the points  $(1, y_i, z_i)$  lie on  $O_\beta$ , we indeed get  $\beta = \gamma k$ . By Lemma 2.9, since  $(O_{\beta k}, O_\beta)$  carries an  $n$ -sided Poncelet Polygon, so does  $(O_{\gamma k}, O_\gamma)$ , which is what we wanted to show.  $\square$



**Corollary 3.6.** *Let  $O_\alpha$  and  $O_\beta$  be conics in  $PG(2, p)$  such that a  $2n$ -sided Poncelet Polygon exists. Then there exists another conic  $O_\gamma$  such that the pair  $(O_\gamma, O_\beta)$  carries an  $n$ -sided Poncelet Polygon.*

*Proof.* Let  $O_\alpha \diamond O_\beta$  such that a  $2n$ -sided Poncelet Polygon can be found and  $\alpha = h\beta$ . This means that

$$(-\beta)(\beta - \alpha) = (-\beta)(\beta - h\beta) = \beta^2(h - 1)$$

is a square in planes  $PG(2, p)$ ,  $p \equiv 3(4)$  and a nonsquare in planes  $PG(2, p)$ ,  $p \equiv 1(4)$ . Since  $\beta^2$  is of course a square, we know whether or not  $(h - 1)$  is a square. To show the statement above, we only have to show that for  $\gamma = k\beta$ :

$$h - 1 \text{ is a square} \Leftrightarrow k - 1 \text{ is a square}$$

To see this, we use our formula for  $2n$ -sided Poncelet Polygons seen in Lemma 3.5:

$$h - 1 = \frac{2}{1 - \frac{1}{\sqrt{k}}} - 1 = \frac{\sqrt{k} + 1}{\sqrt{k} - 1} = \frac{(\sqrt{k} + 1)^2}{k - 1}$$

This gives us:

$$(h - 1)(k - 1) = (\sqrt{k} + 1)^2$$

So we have  $(h - 1)$  a square if and only if  $(k - 1)$  a square.  $\square$

*Example 3.7.* We have already seen in Lemma 3.3 that if  $(O_k, O_1)$  forms a 4-sided Poncelet Polygon, we immediately have  $k = 2$ . Hence by Lemma 3.5, we are able to compute the index  $h$  such that  $(O_h, O_1)$  carries an 8-sided Poncelet Polygon:

$$h = \frac{2}{1 - \frac{1}{\sqrt{k}}} = \frac{2}{1 \pm \frac{1}{\sqrt{2}}} = 4 \pm 2\sqrt{2}$$

This is only well defined if 2 is a square. For this, we use the following result from number theory (see [3]):

$$2 \text{ is a square in } GF(p) \Leftrightarrow p \equiv \pm 1(8) \quad (19)$$

By Poncelet's Theorem, the existence of an 8-gon already implies  $8|(p + 1)$ . Hence, the condition  $p \equiv -1(8)$  is again equivalent to a purely number theoretic result.

The next goal is to deduce such relations for all  $n$ -sided Poncelet Polygons,  $n$  odd. The main idea how to proceed lies already in the following result:

**Lemma 3.8.** *Let  $O_k \diamond O_1$  carry an  $n$ -sided Poncelet Polygon for the points  $B_1, \dots, B_n \in O_1$ ,  $n$  odd. Then  $O_{\frac{k^2}{(k-2)^2}} \diamond O_1$  carries an  $n$ -sided Poncelet Polygon as well, for the same points  $B_1, \dots, B_n \in O_1$ .*

*Proof.* Let  $O_k \diamond O_1$  such that an  $n$ -sided Poncelet Polygon can be found for  $n$  odd. By Lemma 2.13, we have  $B_i + B_{i+1} \in O_k$  for all  $i = 1, \dots, n$  and  $B_i = (1, y_i, z_i)$ , hence:

$$(2, y_i + y_{i+1}, z_i + z_{i+1}) \in O_k \Rightarrow 4 + k(y_i + y_{i+1})^2 + ck(z_i + z_{i+1})^2 = 0$$

Using  $1 + y_i^2 + cz_i^2 = 0$ ,  $\forall B_i \in O_1$ , gives:

$$k = \frac{2}{1 - (y_i y_{i+1} + cz_i z_{i+1})}$$

This implies:

$$\frac{k-2}{k} + y_i(-y_{i+1}) + cz_i(-z_{i+1}) = 0$$

which is equivalent to:

$$\frac{k}{k-2} + \frac{k^2}{(k-2)^2} y_i(-y_{i+1}) + c \frac{k^2}{(k-2)^2} z_i(-z_{i+1}) = 0$$

Again, this relation can be reformulated as an incidence relation:

$$(1, y_i, z_i) \cdot \left( \frac{k}{k-2}, -\frac{k^2}{(k-2)^2} y_{i+1}, -\frac{k^2}{(k-2)^2} z_{i+1} \right) = 0$$

Hence we need

$$(1, y_i, z_i) \in t_{\frac{k^2}{(k-2)^2}} \left( \frac{k}{k-2}, -y_{i+1}, -z_{i+1} \right)$$

as well as

$$(1, y_{i+1}, z_{i+1}) \in t_{\frac{k^2}{(k-2)^2}} \left( \frac{k}{k-2}, -y_i, -z_i \right)$$

Summarizing gives the condition gives:

$$(1, y_{i+1}, z_{i+1}), (1, y_{i-1}, z_{i-1}) \in t_{\frac{k^2}{(k-2)^2}} \left( \frac{k}{k-2}, -y_i, -z_i \right)$$

This can be done for all  $i = 1, \dots, n$  and since  $n$  is odd, for  $O_{\frac{k^2}{(k-2)^2}} \diamond O_1$ , an  $n$ -sided Poncelet Polygon is given via the same points  $B_1, \dots, B_n$ .  $\square$

An immediate corollary using Lemma 2.9 is the following:

**Corollary 3.9.** *The conics  $O_1 \diamond O_{\beta^2}$  carry an  $n$ -sided Poncelet Polygon if and only if  $O_{\frac{1}{\beta^2}} \diamond O_1$  carries an  $n$ -sided Poncelet Polygon.*

*Remark 3.10.* We have seen that for triangles, there is only one conic  $O_k$  such that  $O_k \diamond O_1$  form a 3-sided Poncelet Polygon, namely  $O_4$ . In this case, we should therefore have:

$$k = \frac{k^2}{(k-2)^2}$$

This is equivalent with

$$k^2 - 5k + 4 = 0 \Leftrightarrow k = 1 \text{ or } 4$$

Hence, we obtain again the condition  $k = 4$ , which we already computed in Lemma 3.1 by using other methods.

The procedure shown in the proof above can be iterated. To avoid long expressions, we have:

**Definition 3.1.**  $t_0 := k$ ,  $t_{i+1} := \frac{t_i^2}{(t_i-2)^2}$

Recall that for a given Poncelet  $n$ -gon using the points  $B_1, \dots, B_n$  on  $O_1$  and tangents of some  $O_\alpha$ , there are  $\frac{\phi(n)}{2} - 1$  more conics  $O_\gamma$  such that  $(O_\gamma, O_1)$  carries an  $n$ -sided Poncelet Polygon.

*Example 3.11.* We know that for  $O_\alpha \diamond O_1$  a 5-sided Poncelet Polygon for the same five points  $B_1, \dots, B_5 \in O_\alpha$  can be constructed in two different ways, since  $\frac{\phi(5)}{2} = 2$ . Fix an ordering of the points  $B_1, \dots, B_5$  and start with the following polygon:

$$B_1 \xrightarrow{A_1} B_2 \xrightarrow{A_2} B_3 \xrightarrow{A_3} B_4 \xrightarrow{A_4} B_5 \xrightarrow{A_5} B_1$$

The other 5-gon is then given by connecting  $B_i$  and  $B_{i+2}$ .

$$B_1 \xrightarrow{C_1} B_3 \xrightarrow{C_2} B_5 \xrightarrow{C_3} B_2 \xrightarrow{C_4} B_4 \xrightarrow{C_5} B_1$$

Note that connecting  $B_i$  and  $B_{i+3}$  gives in fact the same polygon again, since we can read the above polygon by reversing the direction.

For 5-sided Poncelet Polygons, we therefore get the following condition:

$$t_0 \neq t_1 \text{ and } t_0 = t_2$$

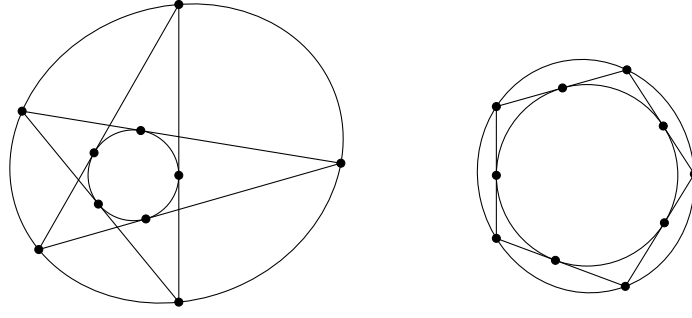


Figure 2: Two different 5-sided Poncelet Polygons can be constructed using the same five points on the outer conic.

We have to solve:

$$k = \frac{k^4}{(k^2 - 2(k-2)^2)^2}$$

which is equivalent to

$$(k-1)(k-4)(16-12k+k^2) = 0$$

We obtain the following four solutions:

$$k \in \{1, 4, 6 + 2\sqrt{5}, 6 - 2\sqrt{5}\}$$

Since  $k = 1$  and  $k = 4$  solves  $t_0 = t_1$ , we find that  $k = 6 \pm 2\sqrt{5}$  implies that if  $O_k \diamond O_1$ , then  $(O_k, O_1)$  carries a 5-sided Poncelet Polygon. A result by Gauss about quadratic residues (see [3]) says:

$$5 \text{ is a square in } \text{GF}(p) \Leftrightarrow p \equiv \pm 1(5) \quad (20)$$

Hence in all planes  $PG(2, p)$ , in which 5 actually divides  $p+1$ , the square root of 5 is well-defined and the indices of the Poncelet 5-gons given by  $6 \pm 2\sqrt{5}$  can be computed.

Finally, we can prove the theorem how to find the indices  $k$ , such that  $(O_k, O_1)$  carries an  $n$ -sided Poncelet Polygon for  $n$  odd.

**Theorem 3.12.** *Let  $n$  be any odd number,  $n \geq 3$ . Then the indices  $k$  such that  $(O_k, O_1)$  carries an  $n$ -sided Poncelet Polygon in a plane  $PG(2, p)$  are given by the solutions of:*

$$t_0 = t_{\frac{\phi(n)}{2}}, \quad t_0 \neq t_i(p) \quad \forall i < \frac{\phi(n)}{2} \quad (21)$$

For a fixed plane  $PG(2, p)$ , these solutions are called Poncelet Coefficients for  $n$ -sided Poncelet Polygons and denoted by  $k_n^i$ ,  $i = 1, \dots, \frac{\phi(n)}{2}$ .

*Proof.* Let  $O_k \diamond O_1$  carry an  $n$ -sided Poncelet Polygon for the points  $B_1, \dots, B_n$ ,  $n$  odd. Let the points be ordered such that for  $O_{t_0} \diamond O_1$ , we have:

$$B_1 \rightarrow B_2 \rightarrow B_3 \rightarrow \dots \rightarrow B_n \rightarrow B_1$$

We have seen in the proof of Lemma 3.8 that the  $n$ -sided Poncelet Polygon of  $O_{t_1} \diamond O_1$  is given by the order:

$$B_1 \rightarrow B_3 \rightarrow B_5 \rightarrow \dots \rightarrow B_n \rightarrow B_2 \rightarrow \dots \rightarrow B_{n-1} \rightarrow B_1$$

Iterating this, we see that the  $n$ -sided Poncelet Polygon given by  $O_{t_i} \diamond O_1$  has the ordering:

$$B_1 \rightarrow B_{1+2^i} \rightarrow B_{1+2*2^i} \rightarrow B_{1+3*2^i} \dots \rightarrow B_1$$

where the indices are taken cyclically. We already know that there are exactly  $\frac{\phi(n)}{2}$  different Poncelet  $n$ -gons. Since we are only working with  $n$  odd, we can apply Fermat's little Theorem (see [3]) and use

$$1 + 2^{\frac{\phi(n)}{2}} \equiv \pm 1(n)$$

This shows directly that for  $O_{t_{\frac{\phi(n)}{2}}}$  we start the polygon by  $B_1 \rightarrow B_2$  or  $B_1 \rightarrow B_n$  and hence is equivalent with the very first  $n$ -gon. To deduce the coefficients  $k$  such that  $(O_k, O_1)$  carries an  $n$ -sided Poncelet Polygon, we therefore indeed have to solve (21).  $\square$

*Remark 3.13.* Note that for some values of  $n$ , the iteration needs fewer steps than  $\frac{\phi(n)}{2}$ , as the order of 2 modulo  $n$  can be smaller than  $\frac{\phi(n)}{2}$ . In these cases, not all indices can be constructed by starting with one Poncelet  $n$ -gon only. Nevertheless, the condition (21) stays the same but the same coefficients could be derived by computing less, i.e.

$$t_0 = t_{s_n}, \quad t_0 \neq t_i(p) \quad \forall i < s_n$$

where

$$s_n := \min \{s | 2^s \equiv \pm 1(n)\}$$

The smallest example for  $\frac{\phi(n)}{2} \neq s_n$  is  $n = 17$ , where we have  $\frac{\phi(17)}{2} = 8$  but  $2^4 \equiv -1(17)$ , i.e.  $s_{17} = 4$ .

*Example 3.14.* We want to deduce the indices  $k$  such that  $O_k \diamond O_1$  carries a 9-sided Poncelet Polygon in  $PG(2, 53)$ . Since  $\frac{\phi(9)}{2} = 3$ , we have to solve:

$$t_0 = t_3, \quad t_1 \neq t_3, \quad t_2 \neq t_3$$

So we need solutions of:

$$t_0 - t_3 = k - \frac{k^8}{(128 + k(-256 + k(160 + (-32 + k)k)))^2} = 0 \quad \text{mod } 53$$

Rewriting this equation, we have to solve:

$$k^8 - k(128 + k(-256 + k(160 + (-32 + k)k)))^2 = 0 \quad \text{mod } 53$$

We obtain the following solutions:

$$k \in \{1, 4, 13, 36, 40\}$$

Since we can exclude the solutions 1 and 4, as they also solve  $t_2 = t_3$ , we deduce that:

$$O_{13} \diamond O_1, \quad O_{36} \diamond O_1, \quad O_{40} \diamond O_1$$

are the pairs of conics in  $PG(2, 53)$  such that a 9-sided Poncelet Polygon can be constructed.

## 3.2 Poncelet Polynomials

We are now able to give an algorithm to deduce for all pairs  $(O_\alpha, O_\beta)$  in  $PG(2, p)$ , whether an  $n$ -sided Poncelet Polygon can be constructed for a given  $n$ . We use the iteration method described before to find polynomials  $P_n(k)$  such that the zeros belong to the coefficients  $k$  of conics  $O_k$ , such that if  $O_k \diamond O_1$ , then  $(O_k, O_1)$  carries an  $n$ -sided Poncelet Polygon. By Lemma 2.9, this gives information about all pairs  $(O_\alpha, O_\beta)$ .

**Definition 3.2.** A polynomial  $P_n$  with integer coefficients is called *Poncelet Polynomial* for  $n$ -sided Poncelet Polygons, if the zeros modulo  $p$  correspond to the coefficients  $k$ , such that  $O_k \diamond O_1$  carries an  $n$ -sided Poncelet Polygon in  $PG(2, p)$ .

*Example 3.15.* We have already seen in Lemma 3.1 that  $P_3(k) = k - 4$  and in Example 3.11 that  $P_5(k) = 16 - 12k + k^2$ .

By Lemma 3.4, we know that all these polynomials  $P_n$  are of degree  $\frac{\phi(n)}{2}$ , as the existence of one conic  $O_k$ , such that  $(O_k, O_1)$  carries an  $n$ -sided Poncelet Polygon in  $PG(2, p)$  leads to  $\frac{\phi(n)}{2}$  such conics  $O_k$ . Until now, we only know how to produce Poncelet Polynomials  $P_n$  for  $n$  odd, but similar to the Poncelet Coefficients  $k$ , a doubling process can be applied for finding  $P_n$  with  $n$  even. Note that to find the coefficients for an odd  $n$ -sided Poncelet Polygon, we look for indices  $k$ , such that:

$$P_n(k) = 0 \text{ in } GF(p)$$

Applying Lemma 3.5 gives:

$$P_{2n}(k) = \frac{(k-2)^{\phi(n)} P_n\left(\frac{k^2}{(k-2)^2}\right)}{P_n(k)}$$

*Example 3.16.* We have  $P_3(k) = -4 + k$  and  $\phi(3) = 2$ . Hence we have to calculate:

$$(k-2)^2 P_3\left(\frac{k^2}{(k-2)^2}\right) = (k-2)^2 \left(-4 + \frac{k^2}{(k-2)^2}\right) = -4(k-2)^2 + k^2 = -(-4+k)(-4+3k)$$

Dividing by  $P_3(k)$  gives  $P_6(k) = 4 - 3k$ .

For the general case, note that for numbers  $n$  and  $m$  which have the same value  $\phi(n) = \phi(m)$ , it has to be checked by hand, which polynomials of degree  $\frac{\phi(n)}{2}$  given by the iteration belong to the  $n$ -gons and which to the  $m$ -gons. For example, the iteration for  $\frac{\phi(n)}{2} = 3$  gives the following polynomial:

$$-(-4+k)(-1+k)k(-64+96k-36k^2+k^3)(-64+80k-24k^2+k^3)$$

Excluding the factors  $(k-4)$  and  $(k-1)$  which already occur at the first iteration, we find checking by hand:

$$P_7(k) = -64 + 80k - 24k^2 + k^3 \text{ and } P_9(k) = -64 + 96k - 36k^2 + k^3$$

With some computational effort, we are now able to create a list of all Poncelet Polynomials  $P_n$  up to a chosen value of  $n$ . Using this list, we can write down an algorithm to find the numbers  $k$ , such that  $(O_k, O_1)$  form an  $n$ -sided Poncelet Polygon, for all  $O_k \diamond O_1$  in a plane  $PG(2, p)$ . Hence, by recollecting the results described in this discussion, we obtain the algorithm we were looking for, which is described in the following corollary:

**Corollary 3.17.** *The following four steps give a complete description of  $n$ -sided Poncelet Polygons for conic pairs  $(O_\alpha, O_\beta)$  in  $PG(2, p)$ .*

1. Deduce all  $n \geq 3$  with  $n|(p+1)$ . For every such  $n$ , calculate  $\frac{\phi(n)}{2}$ , which gives the number of indices  $k$ , such that an  $n$ -sided Poncelet Polygon can be constructed for  $(O_k, O_1)$ .
2. For all values  $n$  obtained in Step 1, look up the Poncelet Polynomial  $P_n$ .
3. For every  $P_n$  deduced in Step 2, solve  $P_n(k) = 0$  modulo  $p$ . This gives the corresponding Poncelet Coefficients  $k$ , such that an  $n$ -sided Poncelet Polygon can be constructed for  $(O_k, O_1)$ .
4. By using the coordinate transformation described in Lemma 2.9, transform the information obtained in Step 3 to all pairs  $O_\alpha \diamond O_\beta$ .

*Example 3.18.* We want to deduce all relations of conic pairs  $O_\alpha \diamond O_\beta$  in the plane  $PG(2, 11)$ . By following the above algorithm, we have:

- Step 1: The values  $n$ , such that an  $n$ -sided Poncelet Polygon can be constructed, are given by  $n = 3, 4, 6, 12$ . Moreover:

$n$	3	4	6	12
$\frac{\phi(n)}{2}$	1	1	1	2

- Step 2: We have the following Poncelet Polynomials:

$$\begin{aligned} P_3(k) &= -4 + k \\ P_4(k) &= -2 + k \\ P_6(k) &= -4 + 3k \\ P_{12}(k) &= -16 + 16k - k^2 \end{aligned}$$

- Step 3: The zeros of the Poncelet Polynomials in  $GF(11)$  are given by:

$$\begin{aligned} P_3(k) &= 0 \Leftrightarrow k = 4 \\ P_4(k) &= 0 \Leftrightarrow k = 2 \\ P_6(k) &= 0 \Leftrightarrow k = 5 \\ P_{12}(k) &= 0 \Leftrightarrow k = 6, 10 \end{aligned}$$

- Step 4: By suitable collinear transformations, we obtain the complete relation table:

	$O_1$	$O_2$	$O_3$	$O_4$	$O_5$	$O_6$	$O_7$	$O_8$	$O_9$	$O_{10}$
$O_1$		12	3			4			6	12
$O_2$	4			12		3	6		12	
$O_3$					6	12	4	12	3	
$O_4$	3	4	6				12	12		
$O_5$	6			3		12		4		12
$O_6$	12		4		12		3			6
$O_7$			12	12				6	4	3
$O_8$		3	12	4	12	6				
$O_9$		12		6	3		12			4
$O_{10}$	12	6			4			3	12	

## 4 Comparison to other methods

### 4.1 Comparison to the Euclidean Plane

Recall that any point on  $O_k : x^2 + ky^2 + ckz^2 = 0$  has a nonzero  $x$ -coordinate. Because of this, we can project these conics on the affine plane by setting  $x = 1$ . Moreover, we can look at real solutions of the equations. In the proof of Poncelet's Theorem for this family of conics, we have seen that there is an affine transformation which maps the whole family into a family of concentric circles. Let us therefore consider pairs of circles in the Euclidean plane, i.e.

$$\begin{aligned} E_1 : x^2 + y^2 &= 1 \\ E_r : x^2 + y^2 &= r^2, r > 1 \end{aligned}$$

We are now trying to find a suitable radius  $r$  for  $E_r$ , such that a regular  $n$ -sided polygon which is inscribed in  $E_r$  and circumscribed about  $E_1$  can be constructed. It is elementary, that one solution to this problem, namely the circumcircle radius  $r$  of a simple, regular  $n$ -sided polygon is given by:

$$r = \frac{1}{\cos(\frac{\pi}{n})}$$

In terms of Poncelet Coefficients as defined for the finite case, this gives:

$$k_n = \frac{1}{\cos^2(\frac{\pi}{n})}$$

*Example 4.1.* The radius  $r$  for a simple, regular 5-gon is therefore given by  $r = \frac{1}{\cos(\frac{\pi}{5})} = -1 + \sqrt{5}$ . Note that  $(-1 + \sqrt{5})^2 = 6 - 2\sqrt{5}$ , which is exactly one of the zeros of the Poncelet Polygon for 5-gons we obtained over finite fields (see Example 3.11). The second radius  $\tilde{r}$ , which corresponds to the complex 5-gon circumscribed about  $E_1$ , can be calculated as well, namely by  $\tilde{r} = \frac{1}{\cos(\frac{2\pi}{5})}$ , which leads to  $\tilde{r} = 1 + \sqrt{5}$ . Hence we obtain  $\tilde{r}^2 = 6 + 2\sqrt{5}$ , which belongs to the second coefficient for 5-gons obtained in the finite case as well.

Now we turn our attention to the formula deduced for the coefficients  $\tilde{k}$  for  $2n$ -sided Poncelet Polygons in Lemma 3.5. For this, note that:

$$\cos^2\left(\frac{\phi}{2}\right) = \frac{1 + \cos(\phi)}{2}$$

Hence we get:

$$\tilde{k} = \frac{1}{\cos^2(\frac{\pi}{2n})} = \frac{2}{1 + \cos(\frac{\pi}{n})} = \frac{2}{1 + \frac{1}{\sqrt{k}}}$$

which is exactly the formula derived for the finite case.

Since there does not exist a radical expression for  $\cos(\frac{\pi}{n})$  for all integers  $n$ , it is convenient to look again at polynomials with roots  $\frac{1}{\cos^2(\frac{k\pi}{n})}$ . These are closely connected to the  $n$ -th cyclotomic polynomials  $\Phi_n(x)$ . Recall, that those polynomials can be written as

$$\Phi_n(x) = \prod_{1 \leq k \leq n, (k,n)=1} (x - e^{\frac{2\pi i k}{n}})$$

It is immediate that the degree of  $\Phi_n$  is  $\phi(n)$ , the Euler totient function. The zeros of  $\Phi_n(x)$  are given by  $e^{\frac{2\pi i k}{n}}$  for  $(k, n) = 1$ . For a zero  $x$  of  $\Phi_n$ , also  $\bar{x} = \frac{1}{x}$  is a zero. Define:

$$q_n(x + \frac{1}{x}) := \Phi_n(x)x^{-\frac{\phi(n)}{2}}$$

The zeros of  $q_n$  are then given by  $2\Re(e^{\frac{2\pi i k}{n}}) = 2\cos(\frac{2\pi k}{n})$ . Next, define

$$r_n(x) := q_n(2x)$$

which has zeros  $\cos(\frac{2\pi k}{n})$ . In the next step, we consider

$$s_n(x) := r_n(2x - 1)$$

which has zeros  $\frac{1 + \cos(\frac{2\pi k}{n})}{2} = \cos^2(\frac{\pi k}{n})$  for  $k = 1, \dots, n-1$ . Finally, consider

$$P_n(x) = x^{\frac{\phi(n)}{2}} s_n\left(\frac{1}{x}\right)$$

with zeros  $\frac{1}{\cos^2(\frac{\pi k}{n})}$ , which is exactly the polynomial we wanted. Summarizing, we have

$$P_n(x) = x^{\frac{\phi(n)}{2}} \Phi_n(z) z^{-\frac{\phi(n)}{2}}, \quad z = \frac{2 - 2\sqrt{1-x}}{x} - 1$$

with zeros  $\frac{1}{\cos^2(\frac{\pi k}{n})}$  for  $(k, n) = 1$ .

*Example 4.2.* For  $n = 5$ , the cyclotomic polynomial is given by  $\Phi_5(x) = x^4 + x^3 + x^2 + x + 1$ , which leads to

$$P_5(x) = 16 - 12x + x^2$$

which indeed has the zeros  $6 + 2\sqrt{5}$  and  $6 - 2\sqrt{5}$ . Note that this is the Poncelet Polynomial for 5-gons derived in the finite case.

## 4.2 Comparison to Cayley's Criterion

The criterion deduced by Cayley in 1853 (see [1]) reads as follows:

**Theorem 4.3.** *Let  $C$  and  $D$  be the matrices corresponding to two conics generally situated in the projective plane. Consider the expansion*

$$\sqrt{\det(tC + D)} = A_0 + A_1t + A_2t^2 + A_3t^3 + \dots$$

*Then an  $n$ -sided Poncelet Polygon with vertices on  $C$  exists if and only if for  $n = 2m + 1$ , we have*

$$\det \begin{pmatrix} A_2 & \dots & A_{m+1} \\ \vdots & \dots & \vdots \\ A_{m+1} & \dots & A_{2m} \end{pmatrix} = 0$$

*and for  $n = 2m$ , we have*

$$\det \begin{pmatrix} A_3 & \dots & A_{m+1} \\ \vdots & \dots & \vdots \\ A_{m+1} & \dots & A_{2m} \end{pmatrix} = 0$$

In the discussion above, we were mainly interested in pairs of conics  $(O_k, O_1)$  with equations

$$O_k : x^2 + ky^2 + ckz^2 = 0$$

$$O_1 : x^2 + y^2 + cz^2 = 0$$

To apply Cayley's criterion, we therefore have to look at the expansion of the square root of

$$\det \begin{pmatrix} t+1 & 0 & 0 \\ 0 & t+k & 0 \\ 0 & 0 & c(t+k) \end{pmatrix}$$

which is given by

$$\sqrt{ck^2} + \frac{(k+2)\sqrt{ck^2}}{2k}t - \frac{(k-4)\sqrt{ck^2}}{8k}t^2 + \frac{(k-2)\sqrt{ck^2}}{16k}t^3 - \frac{(5k-8)\sqrt{ck^2}}{128k}t^4 + O(t)^5$$

*Example 4.4.* The condition for a 3-sided Poncelet Polygon is given by vanishing of the coefficient of  $t^2$  which is  $A_2 = \frac{(k-4)\sqrt{ck^2}}{8k}$ . This expression is zero if and only if  $k-4=0$ , which is exactly the condition derived in Lemma 3.1 for the finite case.

*Example 4.5.* The condition for 5-sided Poncelet Polygons is given by  $A_2A_4 - A_3^2 = 0$ , which is the same as  $\frac{c((k-12)k+16)}{1024} = 0$ . This is equivalent to  $k^2 - 12k + 16 = 0$ , so again, we obtain the same condition as for the finite case (compare to Example 3.11).

## 5 Conclusion and outlook

The most interesting result in this discussion is that in finite geometries, the condition whether we can find Poncelet Polygons for a given pair of conics relies on number theoretic results, in particular on the theory of quadratic residues and polynomials over finite fields. It reveals to be the same criterion as in the Euclidean plane, where the conditions are derived by using tools such as angle and length, hence tools which are a priori not available in finite planes. The main aim is therefore to get a better understanding of this connection and try to extend the results for arbitrary pairs of conics in finite projective coordinate planes and to reveal other connections between the theory of quadratic residues and trigonometry. Moreover, we want to understand how the situation changes when considering planes of prime power order rather than planes over prime fields. Also, by having a closer look at chains of the conics described above, more insight into their cyclomic behavior could be obtained, in particular when considering the cyclic representation of the plane.



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